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Statistical Entropy of 5D Ricci-Flat Space-Time with Generalized Uncertainty Principle Revised

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Abstract Considering corrections to all orders in the Planck length on the quantum state density from generalized uncertainty principle and using that quantum state density to all degrees of freedom including extra dimension, we calculate the statistical entropy of scalar field in the 5D Ricci-flat space-time without any artificial cutoff. Calculation shows that the entropy is the linear sum of the event horizon area and the cosmological horizon area.

Keywords Statistical mechanical entropy · Generalized uncertainty principle · Ricci-flat space-time · Extra dimension

1 Introduction

Bekenstein and Hawking [1, 2] found that the black hole entropy is proportional to the area of the event horizon by comparing black hole physics with thermodynamics. This is one of the most profound discoveries in the modern physics. The statistical origin of black hole entropy become an important question in theoretical physics. A progress has been made by ‘t Hooft [3], whose brick wall model (BWM) is extensively used to calculate the entropy in a variety of black holes. In this model the Bekenstein-Hawking entropy is identified with the statistical-mechanical entropy arising from a thermal bath of quantum fields propagating outside the event horizon. Because the number of quantum states is divergent at the horizon, the entropy is divergent unless introducing a cutoff ε [4–14]. Recently, people found that the idea of minimal length would correct the state density. The minimal length can be

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approached by non-commutative field theory, in which Feynman propagator display an exponential UV cutoff of the form $\exp(-\lambda p^2)$, where the parameter λ is related to the minimal length [15]. At the quantum mechanical level, the essence of the UV finiteness of the Feynman propagator can be also captured by a non-linear dispersion relation $p = f(k)$, where p and k are the momentum and the wave vector of the particle respectively. At the same time, the commutator between the operators $\hat{x}$ and $\hat{p}$ is generalized to $[\hat{x}, \hat{p}] = i\frac{\partial p}{\partial k}$, moreover the usual momentum measure $\prod_{i=1}^{n} dp^i$ is deformed to [16, 17]

$$\prod_{j=1}^{n} dp^j \prod_{i=1}^{n} \frac{\partial k^i}{\partial p^j}. \tag{1}$$

According to the Refs. [15, 18], $\partial p/\partial k$ is equal to $e^{\lambda p^2}$. From this momentum measure, we can easily obtain the number of quantum states in a volume element in phase cell space

$$dn = e^{-\lambda p^2} \frac{d^n x d^n p}{(2\pi)^n}, \tag{2}$$

where $p^2 = p^i p_i$ is the square of momentum.

A static, 3D spherical symmetric line element with a 4D effective cosmological constant takes the form [19, 20]

$$ds^2 = \frac{\Lambda\xi^2}{3}\left[-f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] + d\xi^2, \tag{3}$$

where ξ is an open noncompact extra dimensional coordinate. The metric function is $f(r) = 1 - 2M/r - \Lambda r^2/3$, and M is the black hole's mass on the brane. The part of this metric inside the square bracket is exactly the same line element as the 4D Schwarzschild-de Sitter (SdS) solution, which is bounded by two horizons—event horizon and cosmological horizon. The parameter Λ is an induced cosmological constant, which is reduced from 5D to 4D. The metric (3) is Ricci-flat $R_{AB} = 0$, and there is no cosmological constant in 5D space. So one can actually treat this Λ as a parameter that comes from the fifth dimension.

In this Ricci-flat 5D brane world, the binary Randall-Sundrum type brane system can be constructed via a coordinate transformation $\xi = \sqrt{3/\Lambda}\exp(\sqrt{\Lambda/3}y)$ [21]. Hence, the metric (3) takes a conformal form

$$ds^2 = e^{2\sqrt{\Lambda/3}y}\left[-f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) + dy^2\right], \tag{4}$$

where $0 \le y \le y_1$ is the extra dimensional coordinate and y_1 is essentially a compactification length of extra dimension. In fact, it is a space-time solution intersecting the brane world, which describes a black hole placed on the hypersurface at the fixed extra coordinate. Its horizon is determined by equation $f(r) = 0$. This equation have three roots, two positive rootsr_c, r_e and a negative root $r_o = -(r_c + r_e)$. The two positive roots correspond to cosmological horizon and event horizon respectively. And the negative one has no physical meaning.

Reference [22] calculates statistical entropy using the quantum state density corrected to leading order in the Planck length in the space-time described by the metric (4). Though corrected state density is applied to three ordinary dimensions, the number of quantum states in the extra dimension degree of freedom is computed by Sommerfeld semi-classical quantization. In this paper, we apply the quantum state density corrected to all orders in the Planck length to all degree of freedom including extra dimension.

2 Entropy of Scalar Field in the 5D Ricci-Flat Space-Time

In curved spacetime, the scalar field satisfying the Klein-Gordon equation read

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^A}\left(\sqrt{-g}g^{AB}\frac{\partial\Phi}{\partial x^B}\right)-m^2\Phi=0, \tag{5}$$

where m is the mass of scalar particle. Substituting metric (4) into (5) and using WKB approximation with $\Phi=e^{-i\omega t}e^{iS(r,\theta,\varphi,y)}$, Klein-Gordon equation can be rewritten as

$$-\omega^2 g^{tt}-g^{rr}\left(\frac{\partial S}{\partial r}\right)^2-g^{\theta\theta}\left(\frac{\partial S}{\partial\theta}\right)^2-g^{\varphi\varphi}\left(\frac{\partial S}{\partial\varphi}\right)^2-g^{yy}\left(\frac{\partial S}{\partial y}\right)^2-m^2=0, \tag{6}$$

where

$$\frac{\partial S}{\partial r}=p_r,\quad \frac{\partial S}{\partial\theta}=p_\theta,\quad \frac{\partial S}{\partial\varphi}=p_\varphi, \tag{7}$$

are classical momenta in ordinary dimensions and $\frac{\partial S}{\partial y}=p_y$ is classical momentum in extra dimension. From (6, 7), we can easily obtain

$$p^2=p_ip^i=-\omega^2g^{tt}-m^2=\frac{\omega^2}{f}e^{-2\sqrt{\Lambda/3}y}-m^2, \tag{8}$$

$$p_r^2=\frac{1}{g^{rr}}(-\omega^2g^{tt}-m^2-g^{\theta\theta}p_\theta^2-g^{\varphi\varphi}p_\varphi^2-g^{yy}p_y^2) \tag{9}$$

in order to make sure $p^2>0$, the energy ω must satisfy the condition $\omega\geq m\sqrt{f}e^{\sqrt{\Lambda/3}y}$.

Using (2) the number of quantum states with energy less than ω is given by

$$\begin{aligned}g(\omega)&=\frac{1}{(2\pi)^4}\int e^{-\lambda p^2}drd\theta d\varphi dydp_rdp_\theta dp_\varphi dp_y\\&=\frac{1}{(2\pi)^4}\int e^{-\lambda p^2}drd\theta d\varphi dy\int\frac{2}{\sqrt{g^{rr}}}(-\omega^2g^{tt}-m^2-g^{\theta\theta}p_\theta^2\\&\quad-g^{\varphi\varphi}p_\varphi^2-g^{yy}p_y^2)^{\frac{1}{2}}dp_\theta dp_\varphi dp_y\\&=\frac{2}{3\pi^2}\int e^{-\lambda(\frac{\omega^2}{f}e^{-2\sqrt{\Lambda/3}y}-m^2)}\frac{r^2\left(\frac{\omega^2}{f}e^{-2\sqrt{\Lambda/3}y}-m^2\right)^2}{\sqrt{f}}e^{4\sqrt{\Lambda/3}y}drdy,\end{aligned} \tag{10}$$

where the integral interval for r are $[r_e, r_e+\varepsilon_e]$ and $[r_c-\varepsilon_c, r_c]$ respectively at the event horizon and cosmological horizon, where ε_e and ε_c correspond to two thin layers at vicinity of the event horizon and cosmological horizon respectively whose proper thickness is the minimal length $\sqrt{\frac{e\lambda}{2}}$. The integral interval for y is $[0, y_1]$. In last line of the above equation, since the limit of f is zero at the horizon, $\frac{\omega^2}{f}e^{-2\sqrt{\Lambda/3}y}-m^2$ can be replaced approximately by $\frac{\omega^2}{f}$. Therefore, the number of quantum states with energy less than ω can be rewritten as follows

$$g(\omega)=\frac{2}{3\pi^2}\int e^{-\lambda\frac{\omega^2}{f}e^{-2\sqrt{\Lambda/3}y}}\frac{r^2\omega^4}{f^2\sqrt{f}}drdy. \tag{11}$$

Now, we calculate the integral about r in the (11), rewriting it as follows

$$g(\omega) = -\frac{2}{3\pi^2}\int\left(-\frac{1}{f^2}\right)f' e^{-\lambda\frac{\omega^2}{f}e^{-2\sqrt{\Lambda/3}y}}\frac{r^2}{f'\sqrt{f}}drdy$$
$$= -\frac{2\omega^4}{3\pi^2}\int\frac{r^2}{f'}\frac{1}{\sqrt{f}}e^{-\lambda\frac{\omega^2}{f}e^{-2\sqrt{\Lambda/3}y}}d\left(\frac{1}{f}\right)dy. \tag{12}$$

$\frac{r^2}{f'}$ in the (12) can be expanded to Taylor's series at the event horizon and the cosmological horizon

$$\frac{r^2}{f'} = \frac{r_i^2}{2\kappa_i} + \frac{r_i\kappa_i + M/r_i}{\kappa_i^2}(r - r_i) + \mathrm{O}((r - r_i)^2), \tag{13}$$

where $r_i = r_e$ or r_c, $\kappa_i = \kappa_e$ or κ_c. The function $f(r)$ can be expanded as follows

$$f(r) = f(r_i) + 2\kappa_i(r - r_i) + \mathrm{O}((r - r_i)^2). \tag{14}$$

We note $e^{-2\sqrt{\Lambda/3}y} = Y$, thus (12) becomes

$$g(\omega) = -\frac{2\omega^4}{3\pi^2}\int\frac{r^2}{f'}\frac{1}{\sqrt{f}}e^{-\lambda\frac{\omega^2}{f}e^{-2\sqrt{\Lambda/3}y}}d\left(\frac{1}{f}\right)dy$$
$$= -\frac{\omega^4}{3\pi^2}\frac{r_e^2}{\kappa_e}\int_0^{y_1}dy\int_{\lambda Y\omega^2 f^{-1}(r_e)}^{\lambda Y\omega^2/2\kappa_e\varepsilon_e}\frac{1}{(\lambda Y\omega^2)^{3/2}}\sqrt{\frac{\lambda Y\omega^2}{f}}e^{-\lambda Y\frac{\omega^2}{f}}d\left(\frac{\lambda Y\omega^2}{f}\right) + t.c.$$
$$= -\frac{\omega^4}{3\pi^2}\frac{r_e^2}{\kappa_e}\int_0^{y_1}dy\int_{\infty}^{\lambda Y\omega^2/2\kappa_e\varepsilon_e}\frac{1}{(\lambda Y\omega^2)^{3/2}}\sqrt{x}e^{-x}dx + t.c.$$
$$= \frac{\omega^4}{3\pi^2}\frac{r_e^2}{\kappa_e}\int_0^{y_1}\frac{1}{(\lambda Y\omega^2)^{3/2}}dy\left\{\left[\int_0^{\infty} - \int_0^{\lambda Y\omega^2/2\kappa_e\varepsilon_e}\right]\sqrt{x}e^{-x}dx\right\} + t.c.$$
$$= \frac{\omega^4}{3\pi^2}\frac{r_e^2}{\kappa_e}\int_0^{y_1}\frac{1}{(\lambda Y\omega^2)^{3/2}}dy\left\{\frac{\sqrt{\pi}}{2} - \int_0^{\lambda Y\omega^2/2\kappa_e\varepsilon_e}\sqrt{x}e^{-x}dx\right\} + t.c.$$
$$= \frac{\omega}{3\pi^2}\frac{r_e^2}{\kappa_e}\frac{1}{\lambda^{3/2}}\times\left\{\frac{\sqrt{\pi}}{2}\frac{1}{\sqrt{3\Lambda}}(e^{\sqrt{3\Lambda}y_1} - 1) - \frac{3}{2}\lambda^{\frac{3}{2}}\omega^3\left(\frac{1}{2\kappa_e\varepsilon_e}\right)^{\frac{3}{2}}y_1\right.$$
$$\left. + \sum_{n=1}^{\infty}\frac{(-1)^n(\lambda\omega^2/2\kappa_e\varepsilon_e)^{n+3/2}}{n!(n+3/2)2n\sqrt{\Lambda/3}}(e^{-2n\sqrt{\Lambda/3}y_1} - 1)\right\} + t.c., \tag{15}$$

where $t.c.$ is the term corresponding to the cosmological horizon.

For scalar particle, the free energy is given by

$$F(\beta) = \frac{1}{\beta}\int dg(\omega)\ln(1 - e^{-\beta\omega})$$
$$= -\int_{m\sqrt{f}e^{\sqrt{\Lambda/3}y}}^{\infty}\frac{g(\omega)}{e^{\beta\omega} - 1}d\omega$$
$$\approx -\int_0^{\infty}\frac{g(\omega)}{e^{\beta\omega} - 1}d\omega. \tag{16}$$

The entropy reads

$$
\begin{aligned}
S &= \beta^2 \frac{\partial F}{\partial \beta} \\
&= -\beta^2 \frac{\partial}{\partial \beta} \int_0^\infty \frac{g(\omega)}{e^{\beta\omega} - 1} d\omega \\
&= \frac{1}{3\pi^2} \frac{r_e^2}{\kappa_e} \frac{\beta}{\lambda^{3/2}} \int_0^\infty \frac{xe^x}{(e^x - 1)^2} \times \left\{ \frac{\sqrt{\pi}}{2} \frac{1}{\sqrt{3\Lambda}} (e^{\sqrt{3\Lambda} y_1} - 1) \right. \\
&\quad \left. - \frac{3}{2} \lambda^{\frac{3}{2}} \frac{x^3}{\beta^3} \left(\frac{1}{2\kappa_e \varepsilon_e} \right)^{\frac{3}{2}} y_1 + \sum_{n=1}^{\infty} \frac{(-1)^n (\lambda x^2 / 2\beta^2 \kappa_e \varepsilon_e)^{n+3/2}}{n!(n+3/2) 2n \sqrt{\Lambda/3}} (e^{-2n\sqrt{\Lambda/3} y_1} - 1) \right\} dx + t.c.
\end{aligned} \tag{17}
$$

Relation between ε and λ can be founded as follows [23]

$$
\sqrt{\frac{e\lambda}{2}} = \int_{r_i}^{r_i+\varepsilon} \frac{1}{\sqrt{f}} dr \approx \int_{r_i}^{r_i+\varepsilon} \frac{1}{\sqrt{2\kappa(r - r_i)}} dr = \sqrt{\frac{2\varepsilon}{\kappa_i}}. \tag{18}
$$

Using this relation and $\beta_i = 2\pi/\kappa_i$, (17) can be reduced to

$$
\begin{aligned}
S = \left(\frac{A_e}{6\pi^2 \kappa_e^2 \lambda^{3/2}} + \frac{A_c}{6\pi^2 \kappa_c^2 \lambda^{3/2}} \right) \int_0^\infty \frac{xe^x}{(e^x - 1)^2} \times \left\{ \frac{\sqrt{\pi}}{2} \frac{1}{\sqrt{3\Lambda}} (e^{\sqrt{3\Lambda} y_1} - 1) \right. \\
\left. - \frac{3x^2}{4e\pi^2} y_1 + \sum_{n=1}^{\infty} \frac{(-1)^n (x^2 / 2e\pi^2)^{n+3/2}}{n!(n+3/2) 2n \sqrt{\Lambda/3}} (e^{-2n\sqrt{\Lambda/3} y_1} - 1) \right\} dx,
\end{aligned} \tag{19}
$$

where A_e and A_c are event horizon area and cosmological horizon area respectively. The integral in (19) is a constant which can be calculated numerically if Λ and y_1 are assigned.

3 Conclusion

We have studied the statistical-mechanical entropy arising from the scalar field in the 5D Ricci-flat space-time by applying the corrected state density to all space dimensions including extra dimension. Seeing (19), we can find the entropy of scalar field on the 5D Ricci-flat space-time is the sum of event horizon term and cosmological horizon term. Each term can be rewritten into the form of $A_i/4$ by selecting λ appropriately.

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